By the convolution theorem, the cross-correlation of two time signals,

$$(A \star B)_t = \int A(t' + t) B(t') dt', \tag{1}$$

is a simple multiplication in Fourier space,

$$\mathcal{F}[A \star B] = \hat{A}_{-\omega} \hat{B}_{\omega}. \tag{2}$$

Assume that $A(t)$ and $B(t)$ are real signals. The Fourier transform must satisfy

$$\hat{A}_{-\omega} = \hat{A}^*_{\omega}, \tag{3}$$

and time-symmetrization,

$$A(t) \to \frac{A(t) + A(-t)}{2}, \tag{4}$$

is effectively like replacing the Fourier transform with its real part,

$$\hat{A}_{\omega} \to \mathrm{Re}\hat{A}_{\omega} = \frac{\hat{A}_{\omega} + \hat{A}_{-\omega}}{2}. \tag{5}$$

A relevant physical observable is the dynamical correlation,

$$C_{\omega} = \left\langle \hat{A}_{-\omega} \hat{B}_{\omega} \right\rangle, \tag{6}$$

with brackets denoting an equilibrium average.

Under symmetrization of the time signals $A(t)$ and $B(t)$, the cross correlation becomes:

$$C_{\omega} \to \tilde{C}_{\omega} = \frac{1}{4} \left\langle (\hat{A}_{\omega} + \hat{A}_{-\omega})(\hat{B}_{\omega} + \hat{B}_{-\omega}) \right\rangle. \tag{7}$$

Because of the equilibrium average, the result must be time-translation invariant. In particular, one may average $\tilde{C}_{\omega}$ over all possible time shifts: $A(t) \to A(t+\phi)$ and $B(t) \to B(t+\phi)$. In Fourier space, the time shift introduces a complex phase $A_{\omega} \to e^{-i\omega\phi} A_{\omega}$, and equivalently $A_{-\omega} \to e^{+i\omega\phi} A_{-\omega}$. For each mode ω, it is sufficient to average over a full periodic cycle, $\phi \in [0, 2\pi/\omega]$. Averaging (7) in this way, the result is,

$$\begin{aligned}
\tilde{C}_{\omega} &= \frac{1}{8\pi/\omega} \int_0^{2\pi/\omega} d\phi \left\langle (e^{-i\omega\phi} \hat{A}_{\omega} + e^{+i\omega\phi} \hat{A}_{-\omega})(e^{-i\omega\phi} \hat{B}_{\omega} + e^{+i\omega\phi} \hat{B}_{-\omega}) \right\rangle \\
&= \frac{1}{4} \left\langle \hat{A}_{\omega} \hat{B}_{-\omega} + \hat{A}_{-\omega} \hat{B}_{\omega} \right\rangle + \frac{1}{8\pi/\omega} \int_0^{2\pi/\omega} d\phi \left\langle e^{-2i\omega\phi} \hat{A}_{\omega} \hat{B}_{\omega} + e^{+2i\omega\phi} \hat{A}_{-\omega} \hat{B}_{-\omega} \right\rangle.
\end{aligned} \tag{8}$$

The two remaining integrals over ϕ evaluate exactly to zero, leaving

$$\tilde{C}_{\omega} = \frac{1}{2} \mathrm{Re} \left\langle \hat{A}_{\omega} \hat{B}_{-\omega} \right\rangle = \frac{1}{2} \mathrm{Re}\, C_{\omega}. \tag{9}$$

To summarize: We may sample the *real* part of the dynamical correlation C_ω as

$$\operatorname{Re} C_\omega = 2\tilde{C}_\omega, \tag{10}$$

where $\tilde{C}_\omega$ denotes the circular cross correlation of the time-symmetrized signals as defined in Eqs. (4) and (7). It makes sense that $\tilde{C}_\omega$ cannot capture the imaginary part of C_ω – the effect of time-symmetrization is to enforce $\tilde{C}_\omega = \tilde{C}_{-\omega}$, such that $\tilde{C}_\omega$ can only be real.

In the special case that $A(t) = B(t)$, the desired time correlation $C_\omega = \left\langle \hat{A}_{-\omega} \hat{A}_\omega \right\rangle = C_{-\omega}$ is already real, and we obtain the exact result,

$$C_\omega = 2\tilde{C}_\omega. \tag{11}$$

The result depends crucially on the presence of an equilibrium average, allowing to average over all possible time-translations.

In Sunny applications, we are ultimately interested in expectations of physical observables, which must be real. For example, $\left\langle \hat{S}_x \hat{S}_y + \hat{S}_y \hat{S}_x \right\rangle$ is a good observable, but $\left\langle \hat{S}_x \hat{S}_y \right\rangle$ is not. Therefore the restriction to the real part of the dynamical correlation C_ω does not seem to impose any limitations.

Numerically, we will have observered the time data $A(t)$ and $B(t)$ only over some finite interval, say $t \in [0, T]$ in multiples of a timestep Δt. In principle, the average over $N/\Delta t$ possible timeshifts ϕ could then be performed explicitly, possibly yielding better statistics.