# Generalized Barycentric Coordinates

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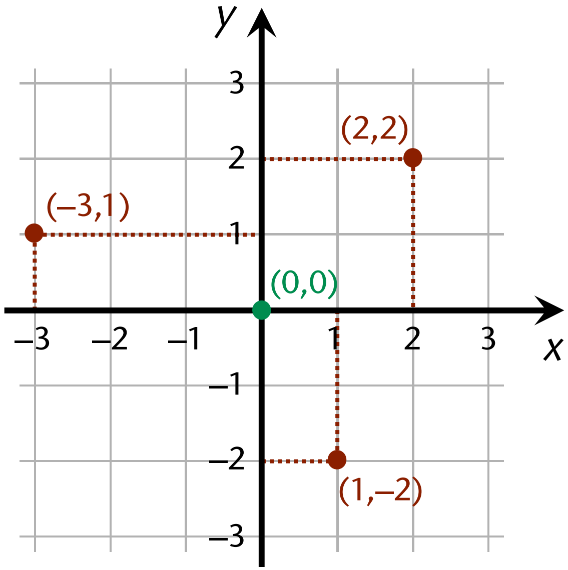
# Cartesian coordinates

*René Descartes (1596–1650)*




point (2,2) with

- $x$-coordinate: 2
- $y$-coordinate: 2

mathematically:

$$(2,2) = 2 \cdot (1,0) + 2 \cdot (0,1)$$

in general:

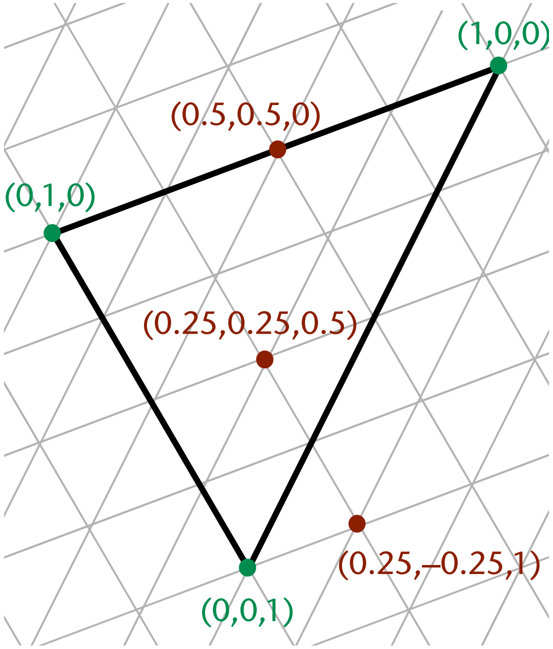
$$(x,y) = x \cdot (1,0) + y \cdot (0,1)$$

$x$- and $y$-coordinates w.r.t. ***base points***

(1,0) and (0,1)

# Barycentric coordinates

*August Ferdinand Möbius*
*(1790–1868)*




point $(a,b,c)$ with 3 coordinates w.r.t. ***base points*** $A, B, C$

mathematically:

$$(a,b,c) = a \cdot A + b \cdot B + c \cdot C$$

where

$$A = (1,0,0)$$
$$B = (0,1,0)$$
$$C = (0,0,1)$$

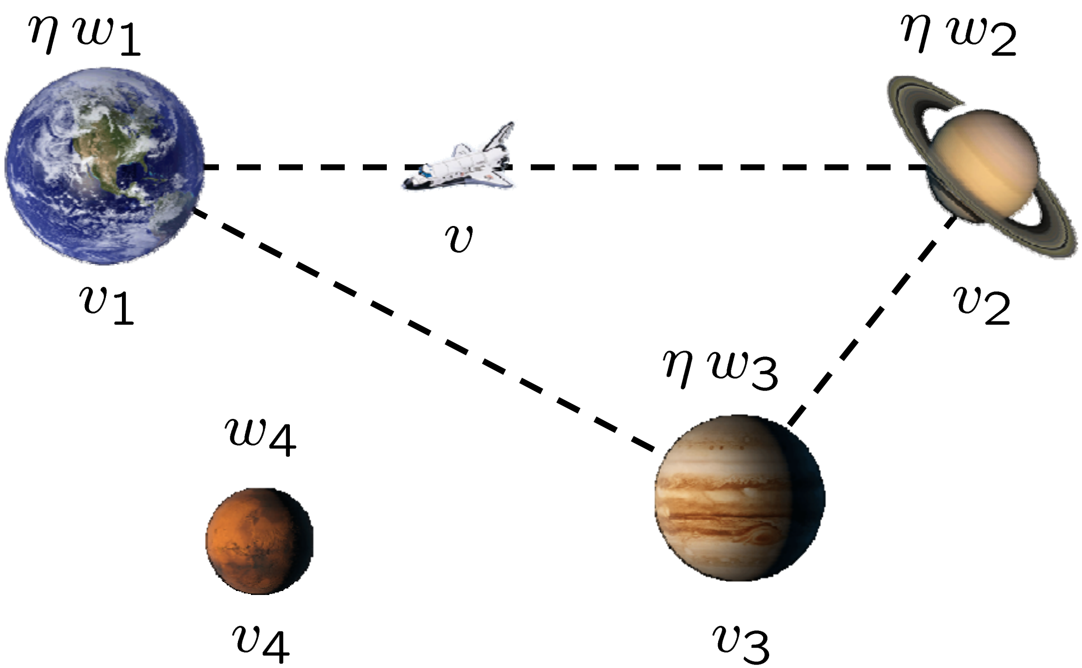
and

$$a + b + c = 1$$

# Barycentric coordinates

- system of masses $w_i$ at positions $v_i$
- position of the system's ***barycentre***:
- $w_i$ are the ***barycentric coordinates*** of $v$
- ***not unique***
- at least $d+1$ points needed to span $\mathbb{R}^d$

$$v = \frac{\sum_i w_i v_i}{\sum_i w_i}$$



# Barycentric coordinates

- **Theorem** [Möbius, 1827] :

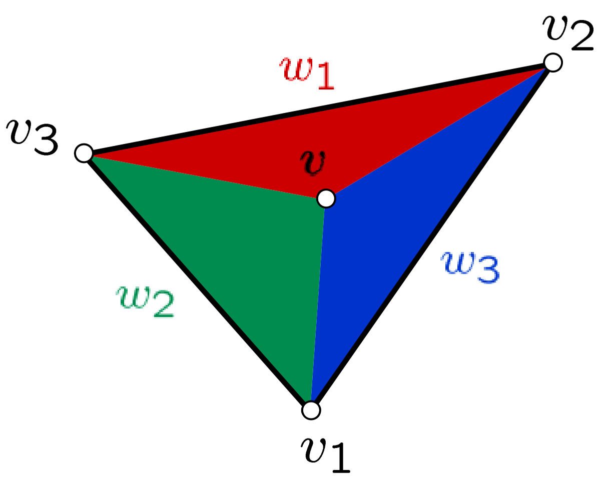
  *The barycentric coordinates* $w_1, \ldots, w_{d+1}$ *of* $v \in \mathbb{R}^d$ *with respect to* $v_1, \ldots, v_{d+1}$ *are* ***unique*** *up to a common factor*

- example: $d = 2$

$$v = \frac{w_1 v_1 + w_2 v_2 + w_3 v_3}{w_1 + w_2 + w_3}$$

$$\Longleftrightarrow$$

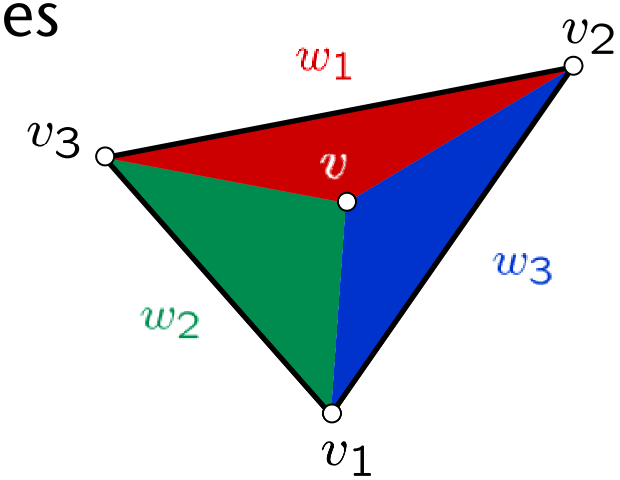
$$w_i = \eta A(v, v_{i+1}, v_{i+2})$$



# Computing areas

- area of triangle $\triangle_1 = [v, v_2, v_3]$ with vertices

$$v = (x, y), \quad v_2 = (x_2, y_2), \quad v_3 = (x_3, y_3)$$




$$w_1 = 2A(v, v_2, v_3) = \det(v_2 - v, v_3 - v)$$
$$= \det \begin{pmatrix} x_2 - x & x_3 - x \\ y_2 - y & y_3 - y \end{pmatrix}$$
$$= (x_2 - x)(y_3 - y) - (x_3 - x)(y_2 - y)$$

- similar for the triangles $\triangle_2 = [v, v_3, v_1]$ and $\triangle_3 = [v, v_1, v_2]$

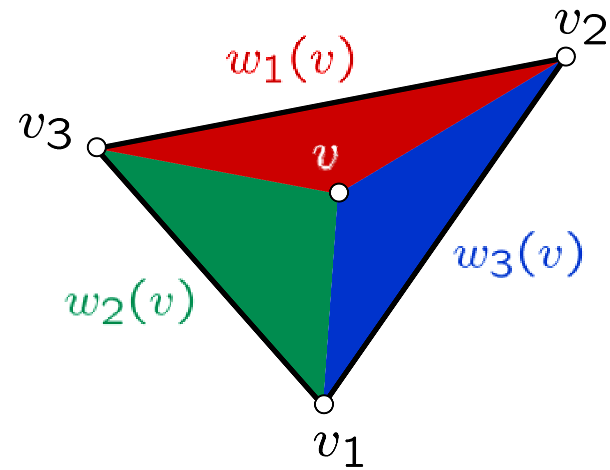
$$w_2 = 2A(v, v_3, v_1) = (x_3 - x)(y_1 - y) - (x_1 - x)(y_3 - y)$$

$$w_3 = 2A(v, v_1, v_2) = (x_1 - x)(y_2 - y) - (x_2 - x)(y_1 - y)$$

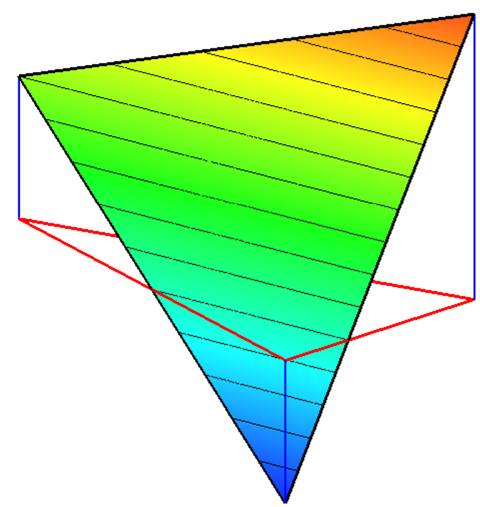
# Barycentric coordinates for triangles

- normalized barycentric coordinates

$$b_i(v) = \frac{w_i(v)}{w_1(v) + w_2(v) + w_3(v)}$$



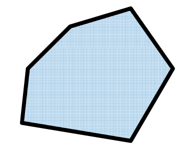

- properties
  - partition of unity $\sum_i b_i(v) = 1$
  - reproduction $\sum_i b_i(v) v_i = v$
  - positivity $b_i(v) > 0, \quad v \in \mathring{\triangle}$
  - Lagrange property $b_i(v_j) = \delta_{ij}$

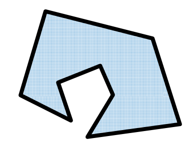


- application
  - linear interpolation of data $F(v) = \sum_{i=1}^{3} b_i(v) f_i$

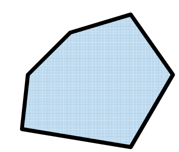
# Generalized barycentric coordinates

- finite-element-method with polygonal elements
  - convex [Wachspress 1975]
  - weakly convex [Malsch & Dasgupta 2004]
  - arbitrary [Sukumar & Malsch 2006]



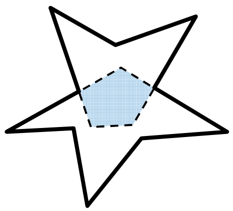


- interpolation of scattered data
  - *natural neighbour interpolants* [Sibson 1980]
  - – " – of higher order [Hiyoshi & Sugihara 2000]
  - Dirichlet tessellations [Farin 1990]

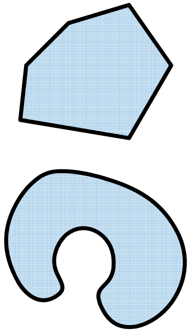


# Generalized barycentric coordinates

- parameterization of piecewise linear surfaces
  - *shape preserving* coordinates [Floater 1997]
  - *discrete harmonic* (DH) coordinates [Eck et al. 1995]
  - *mean value* (MV) coordinates [Floater 2003]



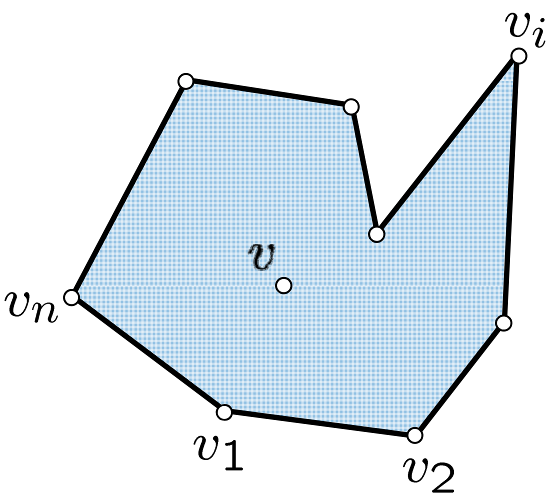
- other applications
  - discrete minimal surfaces [Pinkall & Polthier 1993]
  - colour interpolation [Meyer et al. 2002]
  - boundary value problems [Belyaev 2006]



# Arbitrary polygons

- barycentric coordinates $w_1(v), \ldots, w_n(v)$

$$v = \frac{\sum_{i=1}^n w_i(v) v_i}{\sum_{j=1}^n w_j(v)}$$




- normalized coordinates

$$b_i(v) = \frac{w_i(v)}{\sum_{j=1}^n w_j(v)}$$

- properties
  - partition of unity $\sum_{i=1}^n b_i(v) = 1$
  - reproduction $\sum_{i=1}^n b_i(v) v_i = v$

linear precision

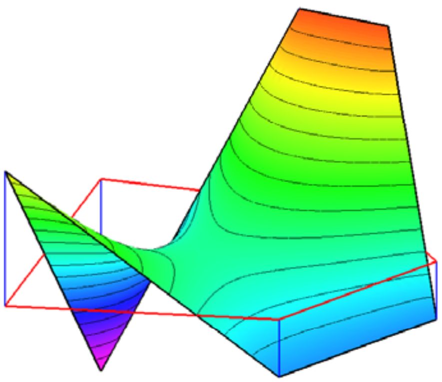
$$\Rightarrow \sum_{i=1}^{n} b_i(v)\phi(v_i) = \phi(v)$$

for all $\phi \in \pi_1$

# Convex polygons

[Floater, H. & Kós 2006]

- **Theorem:** If all $w_i(v) > 0$, then
  - positivity $b_i(v) > 0$
  - Lagrange property $b_i(v_j) = \delta_{ij}$
  - linear along boundary $b_i|_{[v_i, v_{i+1}]} \in \pi_1$

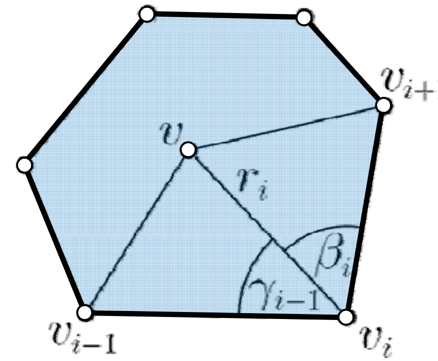


- application
  - interpolation of data given at the vertices $$F(v) = \sum_{i=1}^{n} b_i(v) f_i$$
  - $F(v)$ inside the convex hull of the $f_i$
  - direct and efficient evaluation

# Examples

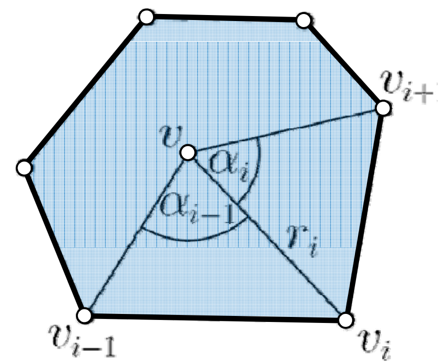
- ***Wachspress*** (WP) coordinates

$$w_i = \frac{\cot \gamma_{i-1} + \cot \beta_i}{{r_i}^2}$$



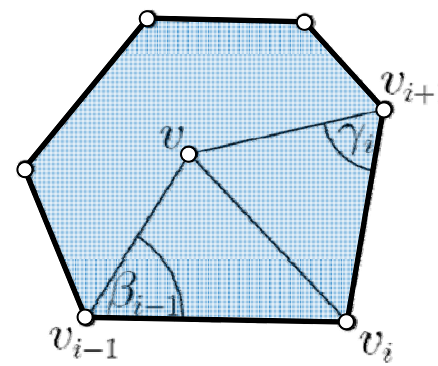

- ***mean value*** (MV) coordinates

$$w_i = \frac{\tan(\alpha_{i-1}/2) + \tan(\alpha_i/2)}{r_i}$$




- ***discrete harmonic*** (DH) coordinates

$$w_i = \cot \beta_{i-1} + \cot \gamma_i$$



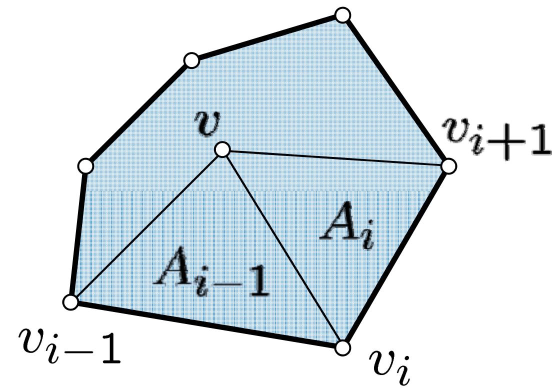
# Normal form

[Floater, H. & Kós 2006]

- **Theorem:** All barycentric coordinates can be written as

$$w_i = \frac{c_{i+1}A_{i-1} - c_i B_i + c_{i-1}A_i}{A_{i-1}A_i}$$

with certain real functions $c_i$

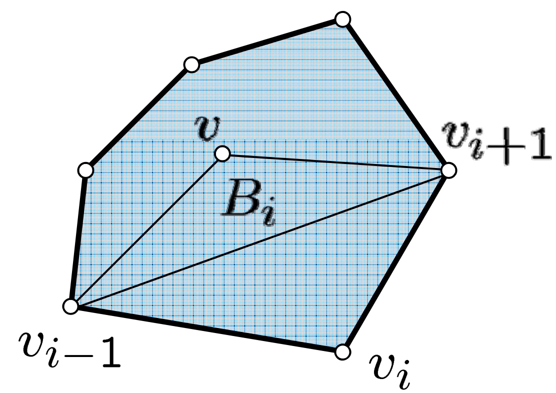



- *three-point coordinates*
  - $c_i = f(r_i)$ with $r_i = \|v - v_i\|$

- **Theorem:** Such a generating function

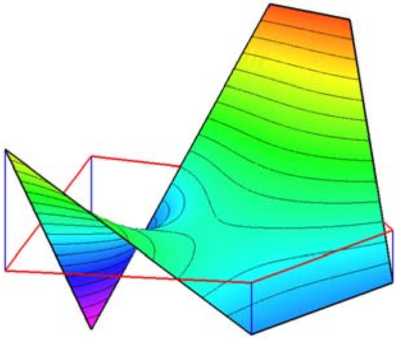
$$f : \mathbb{R}^+ \to \mathbb{R}$$

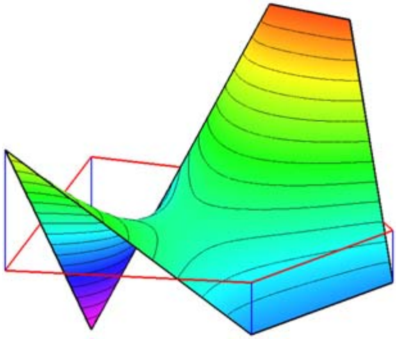
exists for all three-point coordinates

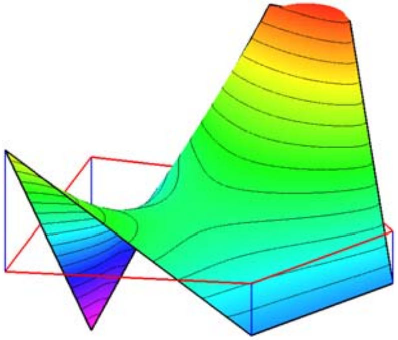


# Three-point coordinates

- **Theorem:** $w_i(v) > 0$ if and only if $f$ is
  - positive $f(r) > 0$
  - monotonic $f'(r) \geq 0$
  - convex $f''(r) \geq 0$
  - sub-linear $f'(r) \leq f(r)/r$
- examples
  - WP coordinates $f(r) = 1$
  - MV coordinates $f(r) = r$
  - DH coordinates $f(r) = r^2$

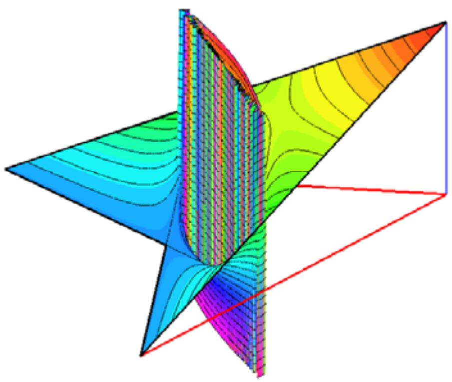






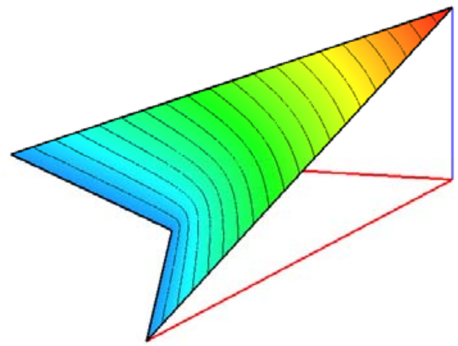
# Non-convex polygons

Wachspress



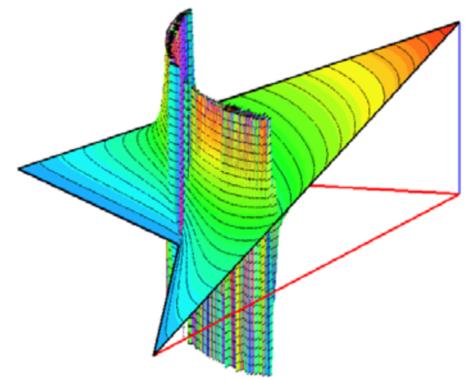
$f(r) = 1$

mean value



$f(r) = r$

discrete harmonic

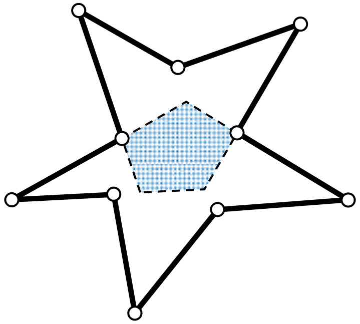


$f(r) = r^2$

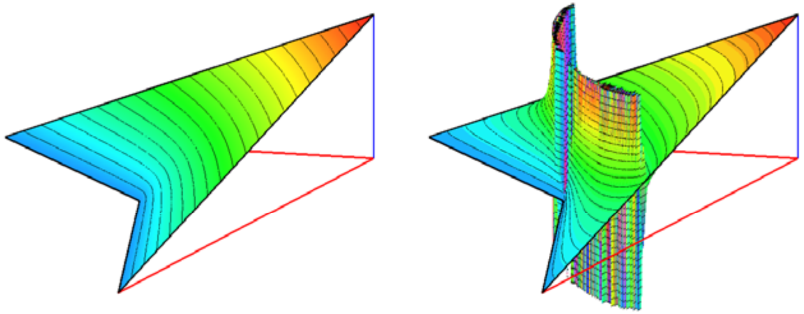
- poles, if $W(v) = \sum_{j=1}^{n} w_j(v) = 0$, because $b_i(v) = \dfrac{w_i(v)}{W(v)}$

# Star-shaped polygons

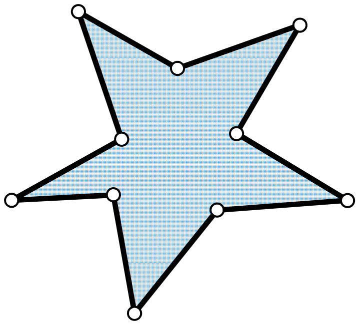
- **Theorem:** $W(v) \neq 0$ if and only if $f$ is
  - positive $f(r) > 0$
  - super-linear $f'(r) \geq f(r)/r$



- examples
  - MV coordinates $f(r) = r$
  - DH coordinates $f(r) = r^2$



- **Theorem:** $W(v) = 0$ for some $v$ if $f$ is
  - strictly super-linear $f'(r) > f(r)/r$

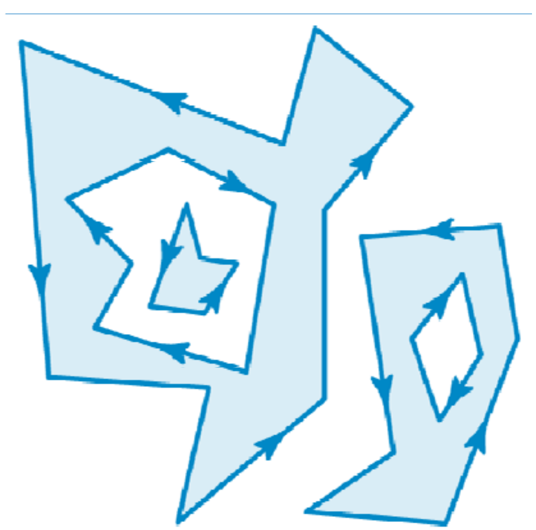


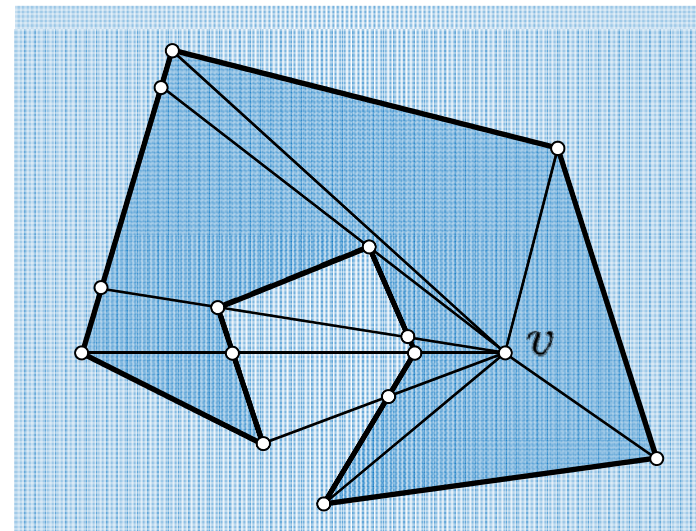
# Mean value coordinates

[H. & Floater 2006]

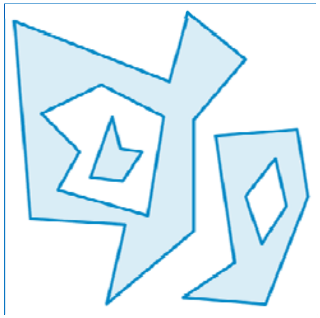
- **Theorem:** MV coordinates have no poles in $\mathbb{R}^2$

$$W(v) = \sum w_j(v) = \sum \kappa_i(v) \neq 0$$





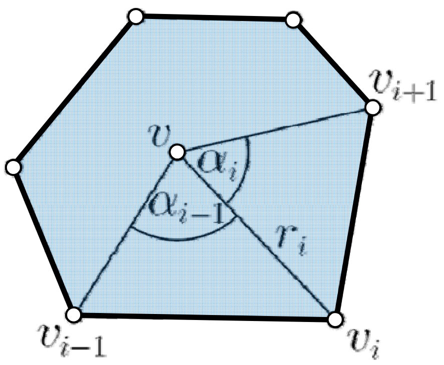
# Mean value coordinates



- properties
  - well-defined everywhere in $\mathbb{R}^2$
  - Lagrange property $b_i(v_j) = \delta_{ij}$
  - linear along boundary $b_i|_{[v_i,v_{i+1}]} \in \pi_1$
  - linear precision $\sum_i b_i(v)\phi(v_i) = \phi(v)$ for $\phi \in \pi_1$
  - smoothness $C^0$ at $v_i$, otherwise $C^\infty$
  - similarity invariance $b_i = \hat{b}_i \circ \psi$ for $\widehat{\Omega} = \psi(\Omega)$
- application
  - direct interpolation of data $F(v) = \sum_{i=1}^{n} b_i(v) f_i$

# Implementation

- Mean Value coordinates




$$w_i = \frac{\tan(\alpha_{i-1}/2) + \tan(\alpha_i/2)}{r_i}$$

$$\tan(\alpha_i/2) = \frac{\sin\alpha_i}{1+\cos\alpha_i} = \frac{r_i r_{i+1}\sin\alpha_i}{r_i r_{i+1} + r_i r_{i+1}\cos\alpha_i}$$

$$= \frac{\det(s_i, s_{i+1})}{r_i r_{i+1} + \langle s_i, s_{i+1}\rangle} = t_i$$

$$s_i = v_i - v$$

$$w_i = \frac{t_{i-1} + t_i}{r_i}$$

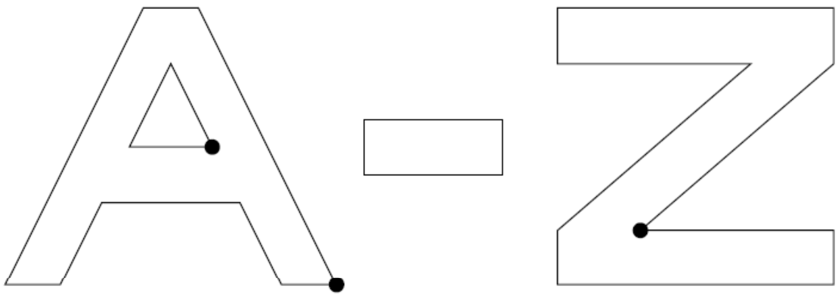
# Implementation

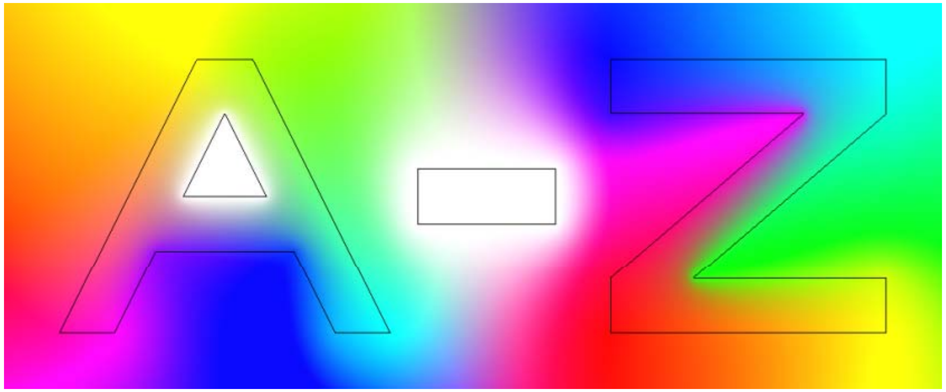
**function** $F(v)$

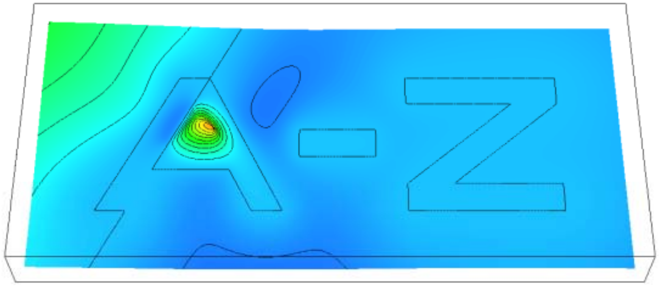
for $i = 1$ to $n$ do
    $s_i := v_i - v$
    $r_i := \|s_i\|$
    if $r_i = 0$ then $\quad // \; v = v_i$
        return $f_i$
for $i = 1$ to $n$ do
    $A_i := \det(s_i, s_{i+1})$
    $D_i := \langle s_i, s_{i+1} \rangle$
    if $A_i = 0$ and $D_i < 0$ then $\quad // \; v \in e_i$
        return $(r_{i+1} f_i + r_i f_{i+1})/(r_i + r_{i+1})$
for $i = 1$ to $n$ do
    $t_i := A_i/(r_i r_{i+1} + D_i)$
$f := 0$
$W := 0$
for $i = 1$ to $n$ do
    $w := (t_{i-1} + t_i)/r_i$
    $f := f + w f_i$
    $W := W + w$
return $f/W$

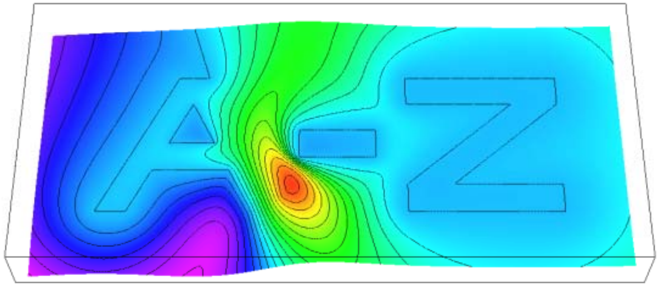
- efficient and robust evaluation of the function $F(v)$

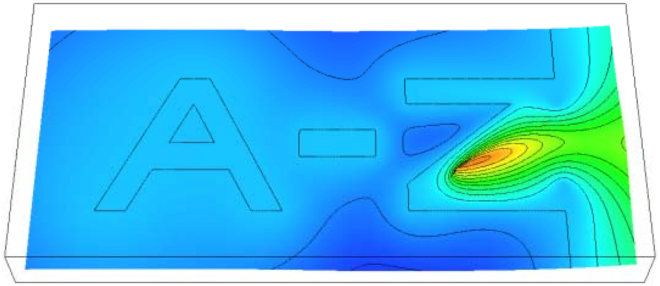
# Colour interpolation



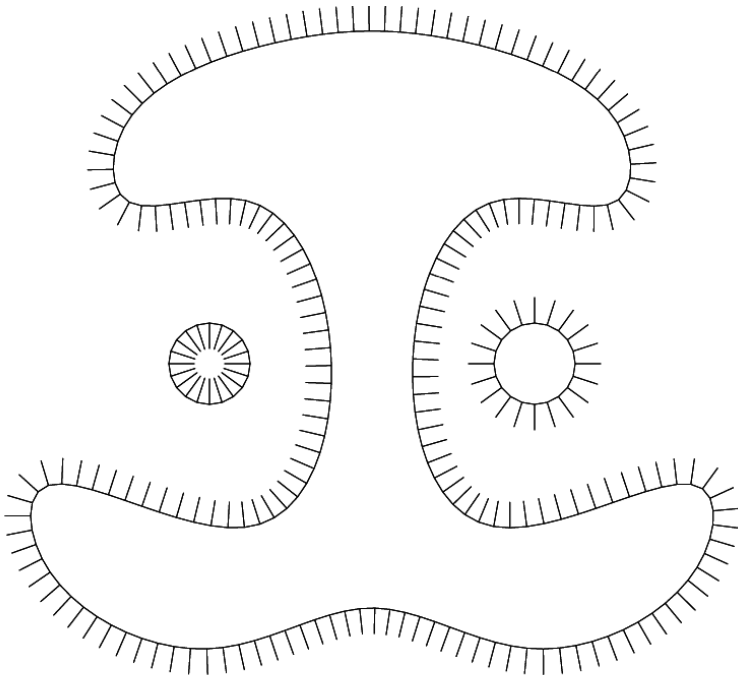



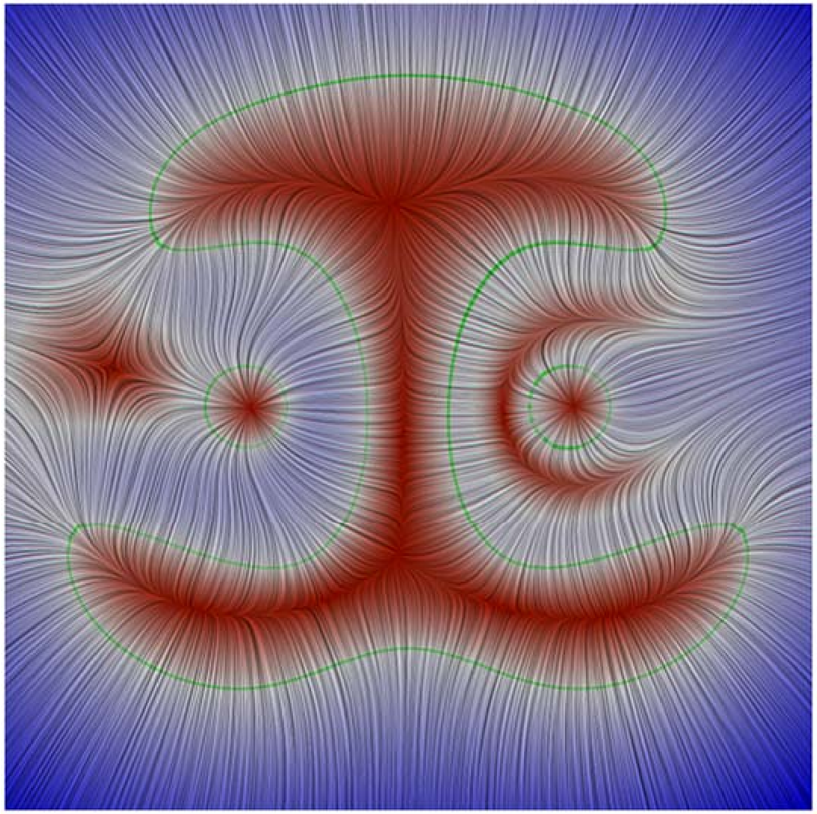




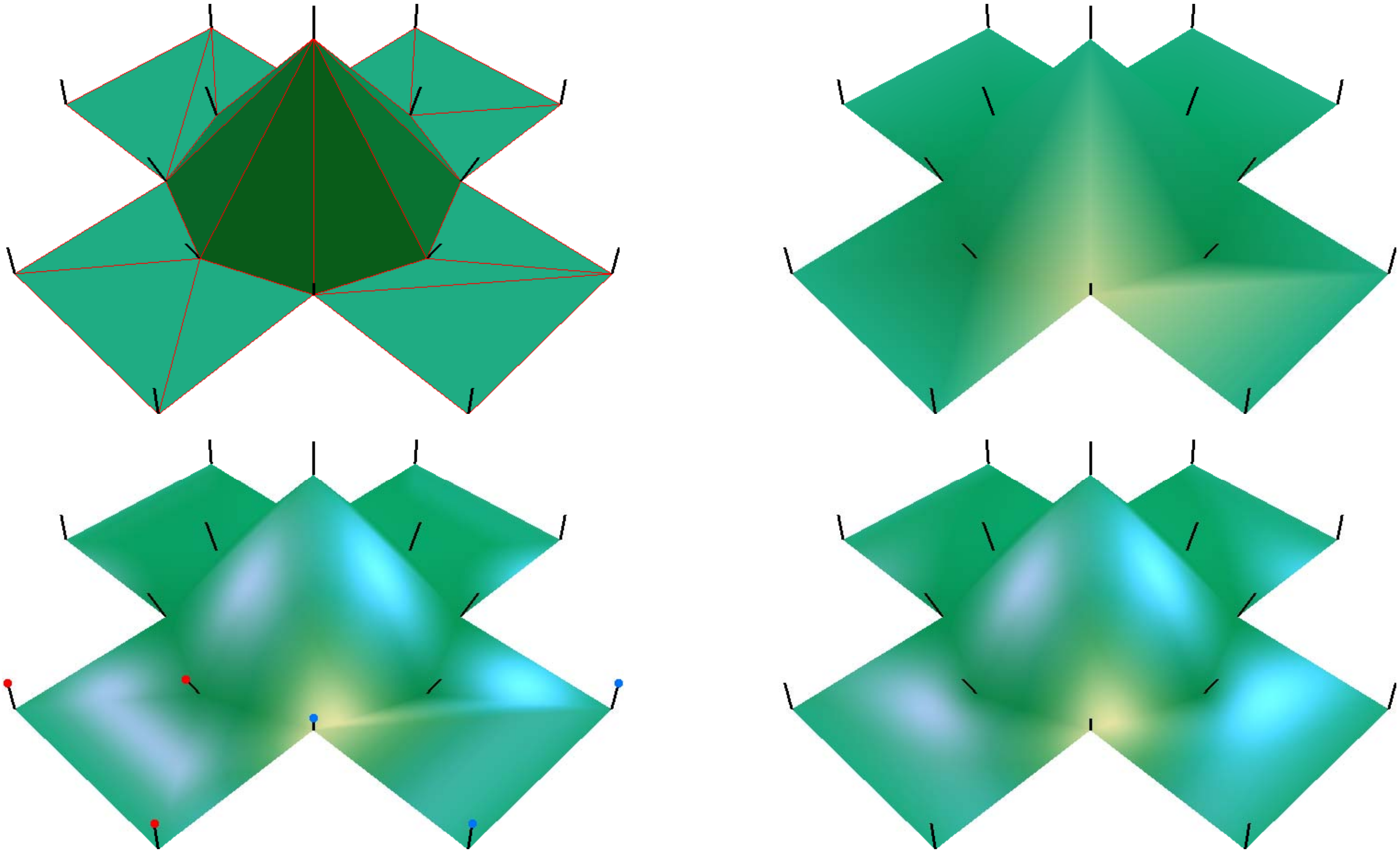


# Vector fields

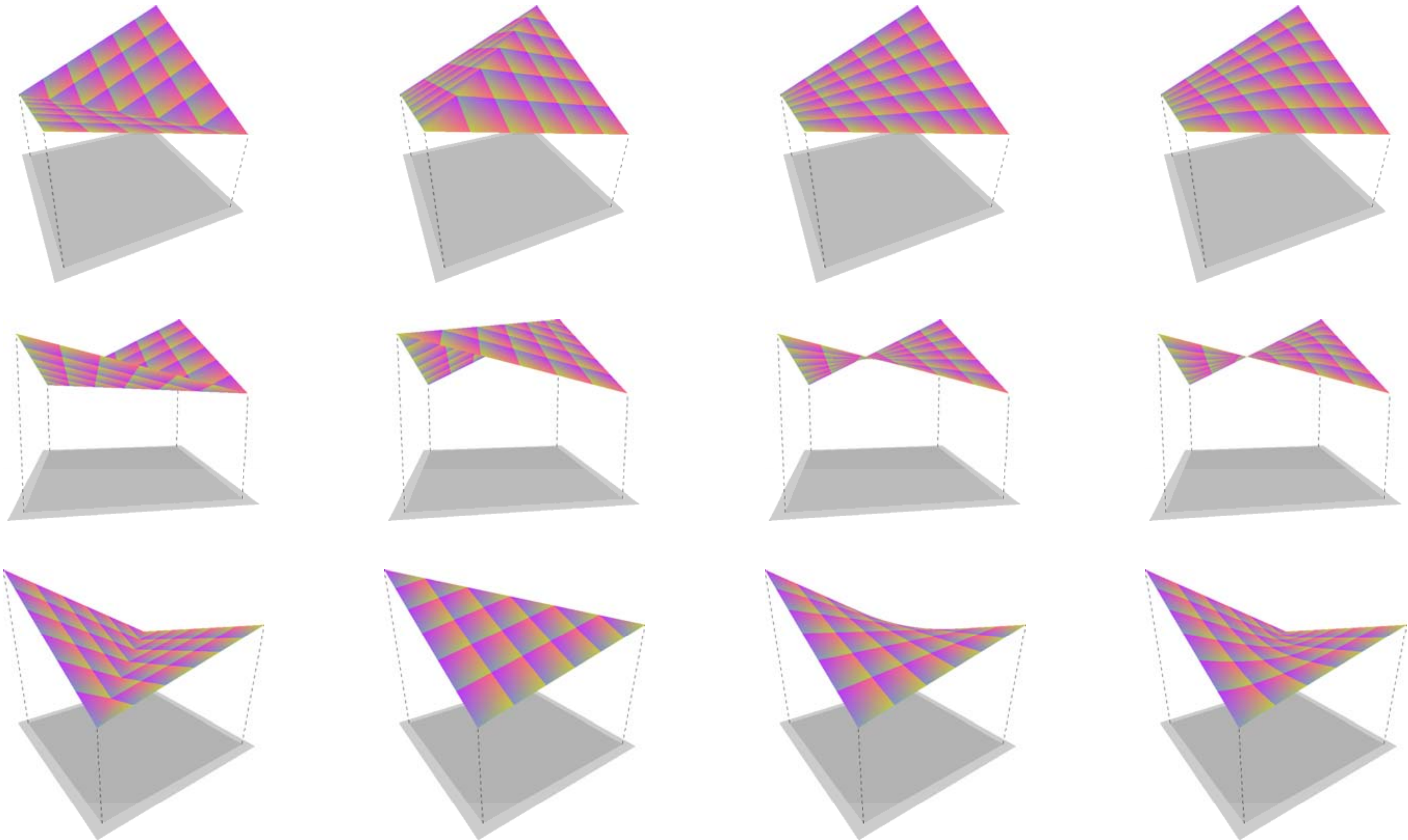




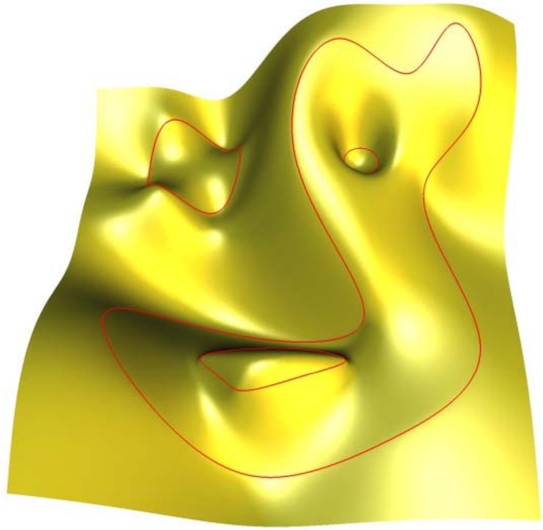
# Smooth shading



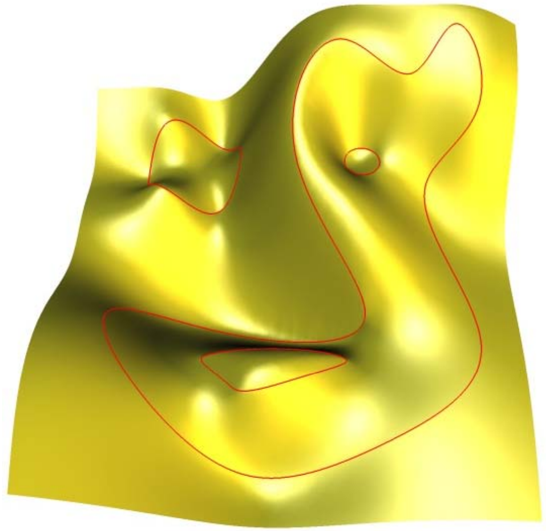
# Rendering of quadrilateral elements



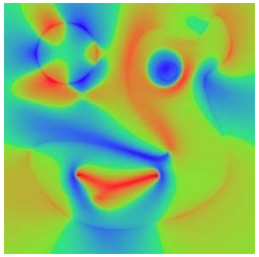
# Transfinite interpolation



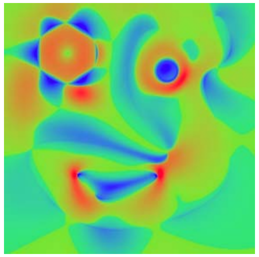
mean value coordinates



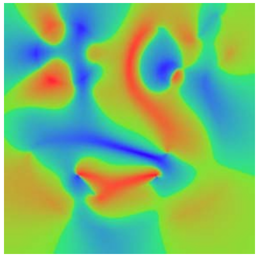
radial basis functions



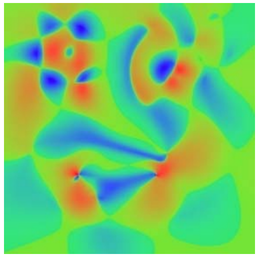
$H$



$K$

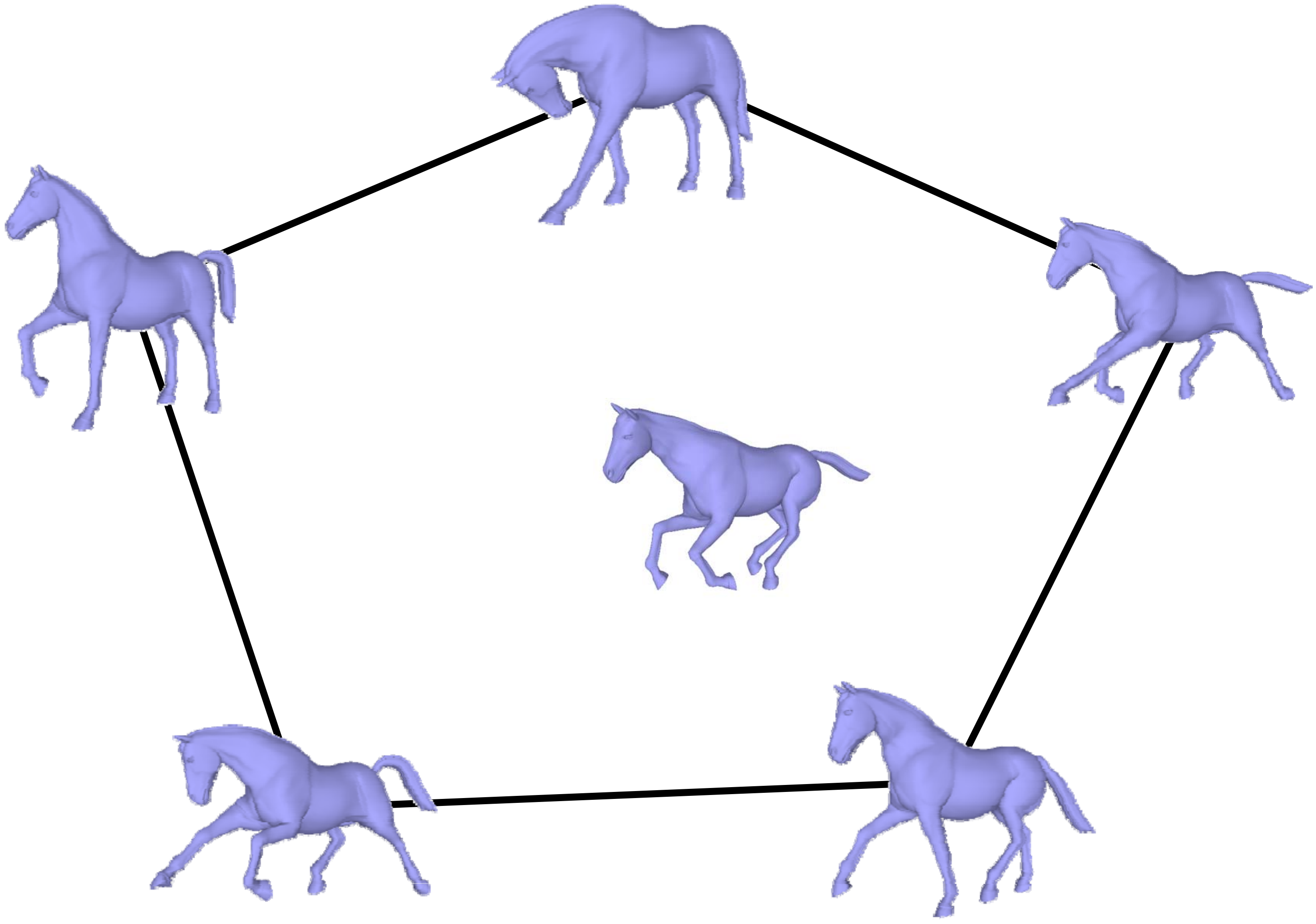


$H$



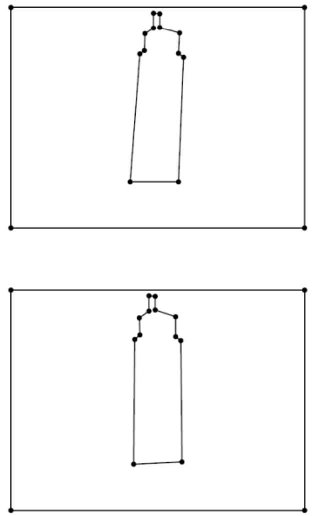
$K$

# Mesh animation



# Image warping

original image



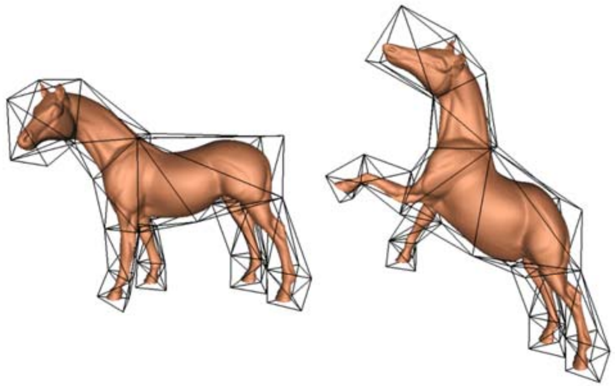
mask

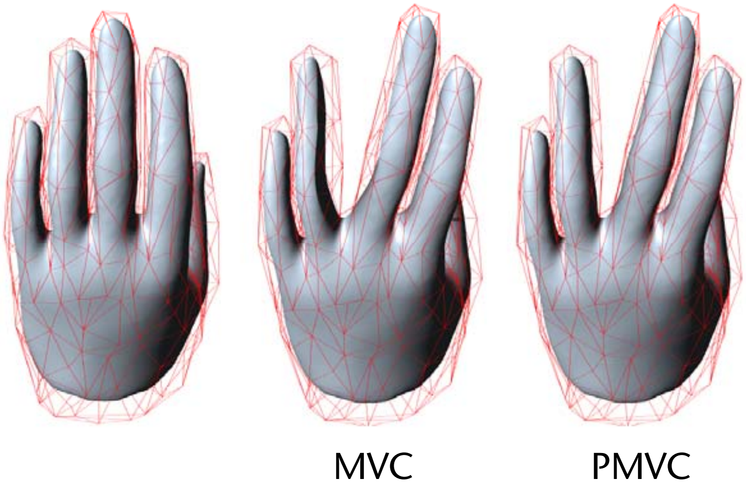
warped image

# Mesh warping

- MV coordinates in 3D

[Ju et al. 2005]






  - negative inside the domain

- positive MV coordinates

[Lipman et al. 2007]

  - only $C^0$-continuous
  - no closed form