Robust Conversion from Matrix to Axis Angle Form

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1 Introduction

Robust conversion from matrix to axis-angle form is not trivial because the transform is plagued with numeric instabilities. In this document we will review the normal form of such a conversion, and provide detailed instructions on how to properly do it.

1.1 Basic Conversion

It is known that the conversion from a SO(3) matrix can be written as

Theorem 1. *The rotation angle θ can be calculated as*

$$\theta = \arccos\left(\frac{\mathrm{tr}(R) - 1}{2}\right) \tag{1}$$

where $\mathrm{tr}(R)$ *is the trace of the rotation matrix.*

As such, the normalized rotation axis is:

Theorem 2. *The normalized rotation axis for a rotation matrix R is*

$$\omega = \frac{\theta}{2\sin(\theta)}\begin{pmatrix} R_{3,2} - R_{2,3} \\ R_{1,3} - R_{3,1} \\ R_{2,1} - R_{1,2} \end{pmatrix} \tag{2}$$

where $R_{i,j}$ are the elements in the matrix.

2 Problem

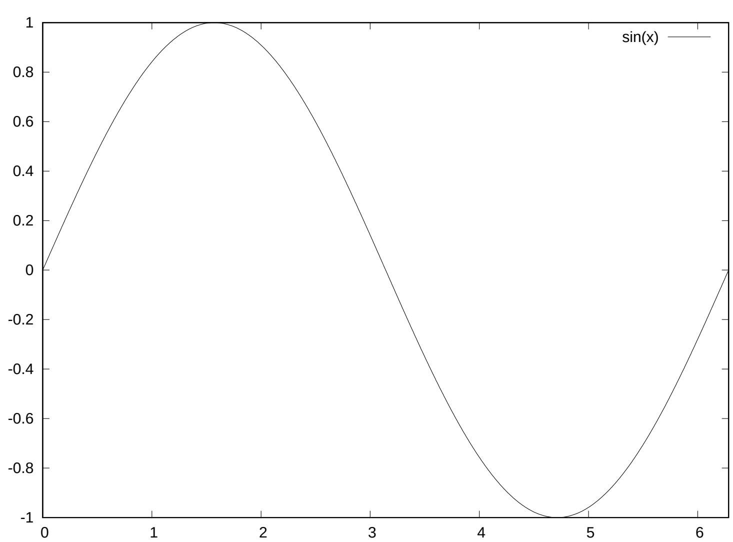

Figure 1. $\sin(x)$

It's pretty straightforward that when $\theta \to 0 + k\pi$, (2) will not work because $\sin(\theta)$ will be 0. From (1) we know that, since $\theta = k\pi$,

$$\begin{aligned} \arccos\left(\frac{\mathrm{tr}(R) - 1}{2}\right) &= k\pi \\ \frac{1}{2}(\mathrm{tr}(R) - 1) &= \pm 1 \\ \mathrm{tr}(R) &= 1 \pm 2 \end{aligned} \tag{3}$$

Thus we know that there are two singular points, −1 and 3, for the axis-angle conversion. Next thing is how to deal with these two points.

3 Properties of the SO(3) transform at the singular points

3.1 $\theta \to 0$

For $\theta = 0$ ($\mathrm{tr}(R) = 3$) the solution is simple: We simply take the Taylor expansion of

$$\frac{\theta}{2\sin(\theta)}$$

at $\theta \to 0$. Note this is not a simple Taylor expansion, but a limit. We can do this in Maxima:

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Maxima 5.45.1 https://maxima.sourceforge.io
using Lisp SBCL 2.1.9
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Dedicated to the memory of William Schelter.
The function bug_report() provides bug reporting information.
```

(%i1) $\theta\text{:}\ \mathrm{acos}\left(\frac{t-1}{2}\right)$

(%o1) $\arccos\left(\frac{t-1}{2}\right)$

(%i2) $\mathrm{mag}\text{:}\ \frac{\theta}{2\sin(\theta)}$

(%o2) $\frac{\arccos\left(\frac{t-1}{2}\right)}{2\sqrt{1-\frac{(t-1)^2}{4}}}$

(%i3) $\mathrm{taylor}(\mathrm{mag}, t, 3, 3)$

(%o3) $\frac{1}{2} - \frac{t-3}{12} + \frac{(t-3)^2}{60} - \frac{(t-3)^3}{280} + \cdots$

Thus the solution around $\theta \to 0$ is

$$\frac{1}{2} - \frac{t-3}{12} + \frac{(t-3)^2}{60} \tag{4}$$

3.2 $\theta \to \pi$

The solution becomes a lot more complicated when at $\theta \to \pi$. This is because at this extreme point you have two solutions: you can go from both left and right of the unit circle!

Note 3. The trace $\mathrm{tr}(R) = -1$ is negative.

In this case we can first convert R to quaternion, which is done by

$$(w, v) = \begin{pmatrix} \frac{1}{2r}(R_{c,b} - R_{b,c}) \\ \frac{1}{2}r \\ \frac{1}{2r}(R_{a,b} + R_{b,a}) \\ \frac{1}{2r}(R_{c,a} + R_{a,c}) \end{pmatrix} \tag{5}$$

where $r := \sqrt{1 + R_{a,a} - R_{b,b} - R_{c,c}}$, $a := \arg\max_{i \in \{1,2,3\}} R_{i,i}$, $b := (a+1) \bmod 3$, $c := (a+2) \bmod 3$.

Normally we want the angle $\theta \in [0, \pi]$, thus we need to make sure that $w > 0$,

and now we can easily get the angle θ as

$$2\,\mathrm{atan2}\left(\sqrt{q_x^2 + q_y^2 + q_z^2}, w\right) \tag{6}$$

and now the magnitude of the vector is

$$|v| := \frac{\theta}{\sin(\theta/2)} \tag{7}$$

However, in GTSAM, we do the following simplification to avoid doing the atan2 (expensive).

First let's set that $q_x^2 + q_y^2 + q_z^2 + w^2 = a^2$. Thus we have (around $w = 0$) this Taylor expansion:

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(%i1) $\theta : 2\,\mathrm{atan2}\left(\sqrt{a^2 - w^2}, w\right)$

(%o1) $2\,\mathrm{atan2}\left(\sqrt{a^2 - w^2}, w\right)$

(%i2) $\mathrm{scale} : \theta / \sin(\theta/2)$

(%o2) $\dfrac{2\,|a|\,\mathrm{atan2}\left(\sqrt{a^2 - w^2}, w\right)}{\sqrt{a^2 - w^2}}$

(%i3) $\mathrm{taylor}(\mathrm{scale}, w, 0, 3)$

(%o3) $\dfrac{\pi\,|a|}{a} - \dfrac{2\,w}{a} + \dfrac{\pi\,w^2}{2\,a\,|a|} - \dfrac{4\,w^3}{3\,a^3} + \cdots$

(%i4)

which indicates that we only need a correction term of $\pi - \frac{2}{a} w$ to get the true magnitude near π.

3.2.1 Verdict

In summary, the logmap (axis-angle form) can be computed as follows at $\mathrm{tr}(R) = -1$.

First, we find the largest diagonal term $a := \arg\max_{i \in \{1,2,3\}} R_{i,i}$. This is required because we are approximately converting the matrix to quaternion form.

Now we calculate the axis-angle in two terms, scale and vector.

The scale is

$$\left(\pi - \frac{2}{a} w\right) \cdot \left(\frac{2}{r}\right) \tag{8}$$

and the vector is

$$v = \begin{pmatrix} r^2 \\ (R_{a,b} + R_{b,a}) \\ (R_{c,a} + R_{a,c}) \end{pmatrix}$$

before permutation. (permutation is [a, a+1 mod 3, a+2 mod 3])

Since we calculate the unnormalized a a' in code:

$$2\,a = \sqrt{4w^2 + (r)^2 + \left(\frac{1}{r}(R_{a,b} + R_{b,a})\right)^2 + \left(\frac{1}{r}(R_{c,a} + R_{a,c})\right)^2}$$

$$2\,r\,a = \sqrt{((R_{c,b} - R_{b,c}))^2 + (2 + 2\,R_{a,a})^2 + ((R_{a,b} + R_{b,a}))^2 + ((R_{c,a} + R_{a,c}))^2} = a'$$

Thus

$$a = a' / (2\,r)$$