Draft version January 7, 2021
Typeset using LaTeX **preprint2** style in AASTeX62

THOR Boundary Layer

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(Dated: December 2020)

1. LOCAL MONIN-OBUKHOV/MIXING LENGTH ALGORITHM FOR THOR

This scheme is based on Holtslag & Boville (1993).

The surface drag coefficients are defined as follows. With the neutral drag coefficient

$$C_N = \left(\frac{k}{\ln\left(\frac{z}{z_r}+1\right)}\right)^2, \tag{1}$$

the coefficients for momentum and heat are

$$C_M = C_N f_M(Ri_0), \tag{2}$$
$$C_H = C_N f_H(Ri_0). \tag{3}$$

The bulk Richardson number of the surface is

$$Ri_0 = \frac{g z_0(\theta_{v,0}-\theta_{v,s})}{\theta_0 |\vec{v}_{h,0}|^2}, \tag{4}$$

where z_0, θ_0, $\vec{v}_{h,0}$ are the height, potential temperature, and horizontal velocity of the lowest layer of the atmosphere. For the moment, we are not considering the effect of condensation, thus we take $\theta_v = \theta$. θ_s is the potential temperature of the surface, and is calculated from the surface temperature and pressure (the latter is extrapolated from the lowest two atmospheric layers).

For stable conditions, the functions f_M and f_H are

$$f_M = f_H = \frac{1}{1+10Ri(1+8Ri)}. \tag{5}$$

For unstable conditions,

$$f_M = 1 - \frac{10Ri}{1+75C_N\sqrt{|Ri|(\frac{z}{z_r}+1)}} \tag{6}$$
$$f_H = 1 - \frac{15Ri}{1+75C_N\sqrt{|Ri|(\frac{z}{z_r}+1)}}. \tag{7}$$

The surface exchange coefficients enter the vertical diffusion scheme via the diffisivities of the lowest interface:

$$K_{M,0} = C_M|\vec{v}_{h,0}|z_0 \tag{8}$$
$$K_{H,0} = C_H|\vec{v}_{h,0}|z_0 \tag{9}$$

At all interfaces above the surface, the diffusitivities are

$$K_M = l_m^2 \left|\frac{\partial \vec{v}_h}{\partial z}\right| f_M^*(Ri) \tag{10}$$
$$K_H = l_h^2 \left|\frac{\partial \vec{v}_h}{\partial z}\right| f_H^*(Ri). \tag{11}$$

The mixing lengths for momentum and heat are

$$l_m = \left(\frac{1}{kz}+\frac{1}{\lambda_m}\right)^{-1}, \tag{12}$$
$$l_h = \left(\frac{1}{kz}+\frac{1}{\lambda_h}\right)^{-1}, \tag{13}$$

where λ_m and λ_h are the asymptotic length scales for heat and momentum. We assume that $\lambda_h = 3\lambda_m$.

In the stable case, $f_M^* = f_H^* = f_M$. In the unstable case,

$$f_M^* = f_H^* = \sqrt{(1-18Ri)}. \tag{14}$$

For each interface the gradient Richardson number is

$$Ri = \frac{g(\partial \theta_v / \partial z)}{\theta |\partial \vec{v}_h / \partial z|^2}. \tag{15}$$

In the case that the shear is zero, we set $Ri = 10^8$. At the top interface of the model, we enforce the boundary condition $K_M = K_H = 0$.

General options necessary for boundary layer:

- `surface = true`
- `Csurf =` (double) [Heat capacity of surface by area (J/K/m^{-2})]

User parameters for boundary layer module:

- `boundary_layer = true | false`
- `bl_type = RayleighHS | LocalMixL`

For `bl_type = RayleighHS`:

- `bl_sigma =` (double) [Rayleigh friction applied at σ levels greater than this]
- `surf_drag =` (double) [Damping coefficient (1/s)]

For `bl_type = LocalMixL`:

- `z_rough =` (double) [Surface roughness length (m)]
- `Ri_crit =` (double) [Critical Richardson number. Currently used only to limit diffusivities above boundary layer]
- `asl_transition_height =` (double) [Height to transition from boundary layer λ_m to free atmosphere λ_m. Can be set to -1 to use boundary layer λ_m throughout. (m)]
- `abl_asym_len =` (double) [Boundary layer λ_m (m)]
- `free_asym_len =` (double) [Free atmosphere λ_m (m)]

2. VERTICAL DIFFUSION ALGORITHM FOR THOR'S BOUNDARY LAYER

2.1. *Heat equation*

The turbulent heat diffusion term has the form

$$\nabla \cdot (\rho K_H \nabla \theta) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \rho K_H \frac{\partial \theta}{\partial r} \right). \tag{16}$$

The horizontal components are neglected. This can be thought of as a source term in the entropy equation,

$$\frac{\partial(\rho\theta)}{\partial t} + \nabla \cdot (\rho \theta \vec{v}) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \rho K_H \frac{\partial \theta}{\partial r} \right). \tag{17}$$

For the purpose of energy conservation, it is better to rewrite this equation in terms of internal energy, $E_{int} = \rho C_V T$. The flux form for the energy evolution (assuming zero velocity, as explained below) should be

$$C_V \frac{\partial \rho T}{\partial t} = \nabla \cdot F \tag{18}$$

where F is the heat flux. We can manipulate entropy equation to produce the correct form for the flux F.

Since the advection term is already incorporated into the dynamical core, we need only to account for the additional effect of the source term on the time derivative, and can temporarily treat the equation as if $\vec{v} = 0$. Using the definition $\theta = T(p_{ref}/p)^\kappa$ and assuming $\vec{v} = 0$, the entropy equation becomes

$$\left(\frac{p_{ref}}{p}\right)^\kappa \frac{C_V}{C_P}\frac{\partial(\rho T)}{\partial t} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \rho K_H \frac{\partial \theta}{\partial r} \right), \tag{19}$$

or

$$\frac{\partial(\rho T)}{\partial t} = \left(\frac{p}{p_{ref}}\right)^\kappa \frac{C_P}{C_V}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \rho K_H \frac{\partial \theta}{\partial r} \right). \tag{20}$$

Now, assuming we discretize this equation carefully, internal energy will be conserved, since $\rho C_V T$ is now our integration variable, as long as C_V is a constant. In order to align with the form of equation 18, we'll take the pressure scaling inside the divergence:

$$\frac{\partial(\rho T)}{\partial t} = \frac{C_P}{C_V}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\left(\frac{p}{p_{ref}}\right)^{\kappa}\rho K_H\frac{\partial\theta}{\partial r}\right). \tag{21}$$

Now we discretize this. For the ith level, the RHS of the equation is

$$\frac{C_P}{C_V}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\left(\frac{p}{p_{ref}}\right)^{\kappa}\rho K_H\frac{\partial\theta}{\partial r}\right) \approx \frac{C_P}{C_V}\frac{1}{r_i^2}\frac{1}{\Delta z_{m,i}}\cdot \left\{\left(r_m^2\left(\frac{p_m^n}{p_{ref}}\right)^{\kappa}\rho_m K_{H,m}\right)_{i+1}\frac{\theta_{i+1}^{n+1}-\theta_i^{n+1}}{\Delta z_{i+1}} - \left(r_m^2\left(\frac{p_m^n}{p_{ref}}\right)^{\kappa}\rho_m K_{H,m}\right)_{i}\frac{\theta_{i}^{n+1}-\theta_{i-1}^{n+1}}{\Delta z_{i}}\right\}, \tag{22}$$

while the LHS is

$$\frac{\partial(\rho T)}{\partial t} \approx \frac{(\rho T)_i^{n+1}-(\rho T)_i^n}{\Delta t}. \tag{23}$$

To make the equation solvable implicitly, we transform back to potential temperature, with pressures defined on the previous time step:

$$\frac{\partial(\rho T)}{\partial t} = \left(\frac{p_i^n}{p_{ref}}\right)^{\kappa}\frac{(\rho\theta)_i^{n+1}-(\rho\theta)_i^n}{\Delta t}. \tag{24}$$

Putting LHS and RHS together:

$$\left(\frac{p_i^n}{p_{ref}}\right)^{\kappa}\frac{(\rho\theta)_i^{n+1}-(\rho\theta)_i^n}{\Delta t} = \frac{C_P}{C_V}\frac{1}{r_i^2}\frac{1}{\Delta z_{m,i}}\cdot \left\{\left(r_m^2\left(\frac{p_m^n}{p_{ref}}\right)^{\kappa}\rho_m K_{H,m}\right)_{i+1}\frac{\theta_{i+1}^{n+1}-\theta_i^{n+1}}{\Delta z_{i+1}} - \left(r_m^2\left(\frac{p_m^n}{p_{ref}}\right)^{\kappa}\rho_m K_{H,m}\right)_{i}\frac{\theta_{i}^{n+1}-\theta_{i-1}^{n+1}}{\Delta z_{i}}\right\}, \tag{25}$$

Quantities with an m subscript are defined at the midpoint/interface between layers, while those without are defined at the center of the layers. Subscripts i, $i+1$, etc., refer to the index of the layer. The n superscript refers to the time step where the terms are to be evaluated, so that we are solving a system of equations for θ_i^{n+1}. The height differences are

$$\Delta z_i = z_i - z_{i-1} \tag{26}$$

$$\Delta z_{i+1} = z_{i+1} - z_i \tag{27}$$

$$\Delta z_{m,i} = z_{m,i+1} - z_{m,i}. \tag{28}$$

The first two are differences between adjacent layer centers, while the third is the difference between the top ($z_{m,i+1}$) and bottom ($z_{m,i}$) of the ith layer. A bit of rearrangement is required to shape this equation in the useful form:

$$a_i\theta_{i-1} + b_i\theta_i + c_i\theta_{i+1} = d_i, \tag{29}$$

which can be solved via Thomas algorithm. In this case, the coefficients are

$$a_i = \frac{C_P}{C_V}\frac{r_{m,i}^2}{r_i^2}\left(\frac{p_{m,i}^n}{p_{ref}}\right)^{\kappa}\frac{\rho_{m,i}K_{H,m,i}}{\Delta z_{m,i}\Delta z_i} \tag{30}$$

$$c_i = \frac{C_P}{C_V}\frac{r_{m,i+1}^2}{r_i^2}\left(\frac{p_{m,i+1}^n}{p_{ref}}\right)^{\kappa}\frac{\rho_{m,i+1}K_{H,m,i+1}}{\Delta z_{m,i}\Delta z_{i+1}} \tag{31}$$

$$b_i = -\left(a_i + c_i + \left(\frac{p_i^n}{p_{ref}}\right)^{\kappa}\frac{\rho_i}{\Delta t}\right) \tag{32}$$

$$d_i = -\left(\frac{p_i^n}{p_{ref}}\right)^{\kappa}\frac{\rho_i}{\Delta t}\theta_i^n. \tag{33}$$

A couple of notes: one, the densities are held constant during this process, so $\rho_i^{n+1} = \rho_i^n$, which is necessary to conserve mass; and two, the form of b_i means that the matrix is always diagonally dominant ($|b_i| > |a_i + c_i|$) as long as $K_H \geq 0$, which should always be the case, so the Thomas algorithm is universally stable.

To update the other thermodynamic variables, we first compute the temperature as

$$T_i^{n+1} = \theta_i^{n+1}\left(\frac{p_i^n}{p_{ref}}\right)^{\kappa}, \tag{34}$$

and then the pressure as

$$p_i^{n+1} = \rho_i R T_i^{n+1}. \tag{35}$$

This ordering of steps is important to preserve energy conservation.

We need an equation to connect to the surface. If written correctly, we can include the surface (as a single layer) in the Thomas algorithm. Conservation of energy for the surface is

$$C_s \frac{\partial T_s}{\partial t} = -F_{sen}, \tag{36}$$

where F_{sen} is the sensible heat flux from surface to atmosphere,

$$F_{sen} = -C_P \left(\frac{p^n_{m,0}}{p_{ref}} \right)^\kappa \rho_{m,0} K_{H,m,0} \frac{\theta^{n+1}_0 - \theta^{n+1}_s}{z_0}, \tag{37}$$

where z_0 is the height of the center of the first layer. In this case, $p_{m,0}$ and $\rho_{m,0}$ are the pressure and density of the atmosphere at the surface, which we calculate by extrapolation from the layers above. Using the assumptions

$$\theta^{n+1}_s = T^{n+1}_s \left(\frac{p_{ref}}{p^n_{m,0}} \right)^\kappa, \tag{38}$$

$$\theta^n_s = T^n_s \left(\frac{p_{ref}}{p^n_{m,0}} \right)^\kappa, \tag{39}$$

we can write the equation for the surface as

$$\begin{aligned} C_s \left(\frac{p^n_{m,0}}{p_{ref}} \right)^\kappa \frac{\theta^{n+1}_s - \theta^n_s}{\Delta t} = \\ C_P \left(\frac{p^n_{m,0}}{p_{ref}} \right)^\kappa \rho_{m,0} K_{H,m,0} \frac{\theta^{n+1}_0 - \theta^{n+1}_s}{z_0}. \end{aligned} \tag{40}$$

The pressure scale factor can be canceled, since this is calculated at same location and time on each side of the equation. For the surface, the coefficients for the Thomas algorithm are

$$a_s = 0 \tag{41}$$

$$c_s = \frac{C_P}{C_s} \frac{\rho_{m,0} K_{H,m,0}}{z_0} \tag{42}$$

$$b_s = -\left(c_s + \frac{1}{\Delta t} \right) \tag{43}$$

$$d_s = -\frac{\theta^n_s}{\Delta t}. \tag{44}$$

Additionally, the a_0 coefficient for the first layer should be calculated with $\Delta z_i = z_0$. The top boundary condition is accounted for in the diffusivity: with j as the top-most level of the boundary layer, $K_{H,m,j+1} = 0$, such that $c_j = 0$.

2.2. *Momentum equation*

Turbulent diffusion is only applied to the horizontal momentum. There are three equations, corresponding to the x, y, z components of $\vec{v}_h$.

$$\nabla \cdot (\rho K_M \nabla \vec{v}_h) = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \rho K_M \frac{\partial \vec{v}_h}{\partial r} \right). \tag{45}$$

Thus the derivatives of the horizontal momentum are given according to

$$\frac{\partial (\rho \vec{v}_h)}{\partial t} = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \rho K_M \frac{\partial \vec{v}_h}{\partial r} \right). \tag{46}$$

The flux form above is used to best conserve momentum in the finite volume approximation. We discretize these equations directly as:

$$\begin{aligned} \frac{(\rho \vec{v}_h)^{n+1}_i - (\rho \vec{v}_h)^n_i}{\Delta t} = \frac{1}{r^2_i} \frac{1}{\Delta z_{m,i}} \cdot \\ \left\{ \left(r^2_m \rho_m K_{M,m} \right)_{i+1} \frac{\vec{v}_{h_{i+1}}^{n+1} - \vec{v}_{h_i}^{n+1}}{\Delta z_{i+1}} - \right. \\ \left. \left(r^2_m \rho_m K_{M,m} \right)_i \frac{\vec{v}_{h_i}^{n+1} - \vec{v}_{h_{i-1}}^{n+1}}{\Delta z_i} \right\}, \end{aligned} \tag{47}$$

Each component is then solved via Thomas algorithm, similar to the heat equation in the previous section. The coefficients are

$$a_i = \frac{r^2_{m,i}}{r^2_i} \frac{\rho_{m,i} K_{M,m,i}}{\Delta z_{m,i} \Delta z_i} \tag{48}$$

$$c_i = \frac{r^2_{m,i+1}}{r^2_i} \frac{\rho_{m,i+1} K_{M,m,i+1}}{\Delta z_{m,i} \Delta z_{i+1}} \tag{49}$$

$$b_i = -\left(a_i + c_i + \frac{\rho_i}{\Delta t} \right) \tag{50}$$

$$\vec{d_i} = -\frac{\rho_i}{\Delta t} \vec{v}_{h_i}^n. \tag{51}$$

Note that $\vec{d_i}$ is a vector with three values, one for each component of $\vec{v}_h$.

We enforce the boundary condition $\vec{v}_h = 0$ at the surface. At the top of the atmosphere, we enforce $K_M = 0$ (values of K_M should always be small above the boundary layer, but this ensures that there is no momentum leak).

3. VERTICAL GRID REFINEMENT FOR THE LOWER ATMOSPHERE

The vertical grid in the lower atmosphere needs to be spaced more finely to resolve the boundary layer. In THOR, the grid spacing can be set according to the soft–plus function. This gives exponential spacing in the lower atmosphere which transitions to linear spacing in the upper atmosphere.

The user control parameters are the height of the top of the lowest layer, z_0, the height of the transition from exponential to linear, z_{tr}, and the height of the model top, z_{top}. From these, we compute an amplitude, sharpness, and vertical offset,

$$\alpha = \frac{z_{tr}}{\ln(2)} \tag{52}$$

$$k_{sh} = \frac{1}{1/n_v - 1} \ln\left(\frac{e^{z_0/\alpha} - 1}{e^{z_{top}/\alpha} - 1}\right) \tag{53}$$

$$x_{tr} = \frac{-\ln(e^{z_0/\alpha} - 1)}{k_{sh}} + \frac{1}{n_v}, \tag{54}$$

where n_v is the total number of vertical levels.

The control volume interfaces are set according to

$$z_{m,i} = \alpha \log(1 + e^{k_{sh}(x_i - x_{tr})}), \tag{55}$$

where $x_i = i/n_v$ for $i = 0$ to n_v. The control volume centers are then defined as the mid-point between each interface

$$z_{c,i} = \frac{1}{2}(z_{m,i} + z_{m,i+1}). \tag{56}$$

User parameters:

- `vert_refined = true | false`
- `lowest_layer_thickness =` (double) [z_0 (m)]
- `transition_altitude =` (double) [z_{tr} (m)]

REFERENCES

Holtslag, A. A. M., & Boville, B. A. 1993, Journal of Climate, 6, 1825, doi: 10.1175/1520-0442(1993)006⟨1825:LVNBLD⟩2.0.CO;2