

Numerical Derivatives

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1 Numerical Accuracy of Expressions

We have to be very careful in these math. For expression A

$$A = \frac{\sin(\theta)}{\theta} \quad (1)$$

The cutoff should be 1×10^{-3} to use the Taylor expansion

```
(%i 1) y:sin(x)/x
```

```
(%o1)  $\frac{\sin(x)}{x}$ 
```

```
(%i 2) taylor(y,x,0,3)
```

```
(%o2)  $1 - \frac{x^2}{6} + \dots$ 
```

For expression B

$$B = \frac{1 - \cos(\theta)}{\theta^2} \quad (2)$$

```
(%i 9) y(x) :=  $\frac{1 - \cos(x)}{x^2}$ 
```

```
(%o9)  $y(x) := \frac{1 - \cos(x)}{x^2}$ 
```

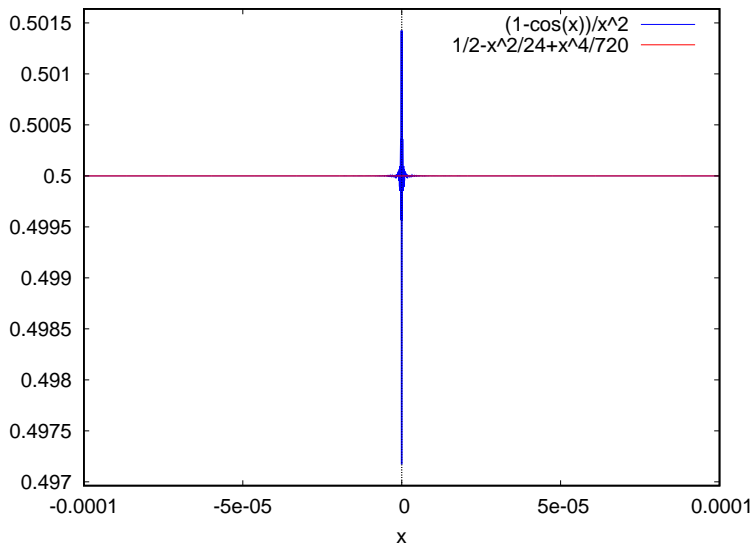
```
(%i 12) taylor(y(x),x,0,4)
```

```
(%o12)  $\frac{1}{2} - \frac{x^2}{24} + \frac{x^4}{720} + \dots$ 
```

```
(%i 13) tm_plot2d([y(x),%],[x,-1e-4,1e-4])
```

```
expt: undefined: 0 to a negative exponent.
```

```
plot2d: expression evaluates to non-numeric value somewhere in plotting range.
```



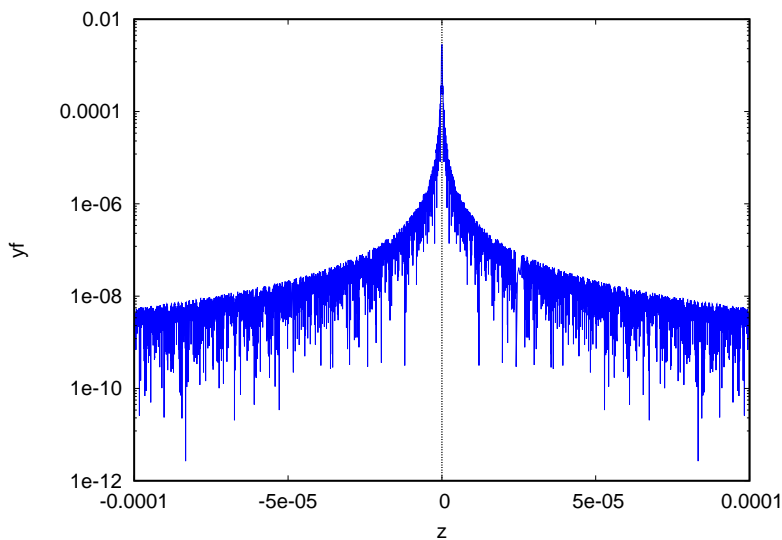
(%o13) true

(%i 16) `yf(z):=abs(float(y(z))-ev(''(taylor(y(x),x,0,4)),numer,x=z))`

(%o16) $yf(z) := \left| \text{float}(y(z)) - \text{ev}\left(\frac{1}{2} - \frac{x^2}{24} + \frac{x^4}{720} + \dots, \text{numer}, x = z\right) \right|$

(%i 17) `tm_plot2d([yf],[z,-1e-4,1e-4],logy)`

arithmetic error FLOATING-POINT-INVALID-OPERATION signalled
 plot2d: expression evaluates to non-numeric value somewhere in plotting range.



(%o17) true

For expression C

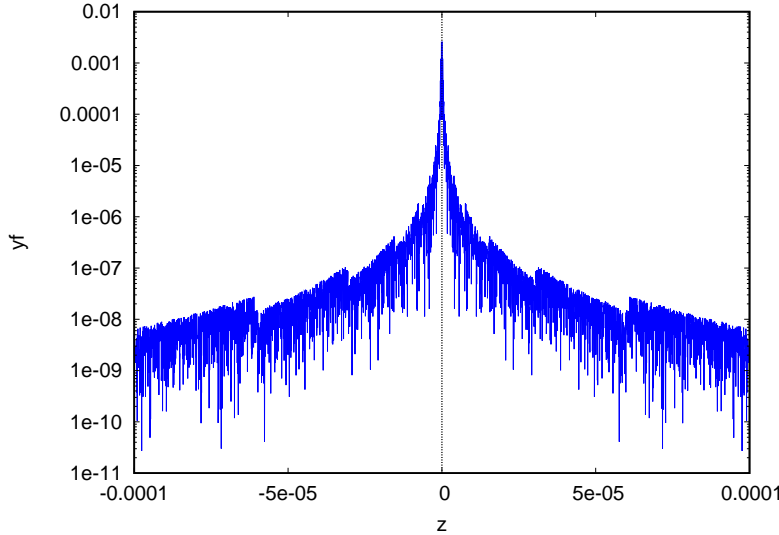
$$C = \frac{\theta - \sin(\theta)}{\theta^3} \quad (3)$$

(%i 18) `y(x) := (x-sin(x))/x^3`

(%o18) $y(x) := \frac{x - \sin(x)}{x^3}$

```
(%i 19) taylor(y(x),x,0,4)
(%o19)  $\frac{1}{6} - \frac{x^2}{120} + \frac{x^4}{5040} + \dots$ 
(%i 20) yf(z):=abs(float(y(z))-ev(''(taylor(y(x),x,0,4)),numer,x=z))
(%o20)  $yf(z) := \left| \text{float}(y(z)) - \text{ev}\left(\frac{1}{6} - \frac{x^2}{120} + \frac{x^4}{5040} + \dots, \text{numer}, x = z\right) \right|$ 
(%i 21) tm_plot2d([yf],[z,-1e-4,1e-4],logy)
```

arithmetic error FLOATING-POINT-INVALID-OPERATION signalled
plot2d: expression evaluates to non-numeric value somewhere in plotting range.



(%o21) true

For expression D

$$D = \frac{A - 2B}{\theta^2} = \frac{\theta \sin(\theta) - 2(1 - \cos(\theta))}{\theta^4} \quad (4)$$

```
(%i 22) y(x) :=  $\frac{\sin(x) - 2\frac{1 - \cos(x)}{x^2}}{x^2}$ 
```

```
(%o22)  $y(x) := \frac{\sin(x) - 2\left(\frac{1 - \cos(x)}{x^2}\right)}{x^2}$ 
```

```
(%i 23) taylor(y(x),x,0,4)
```

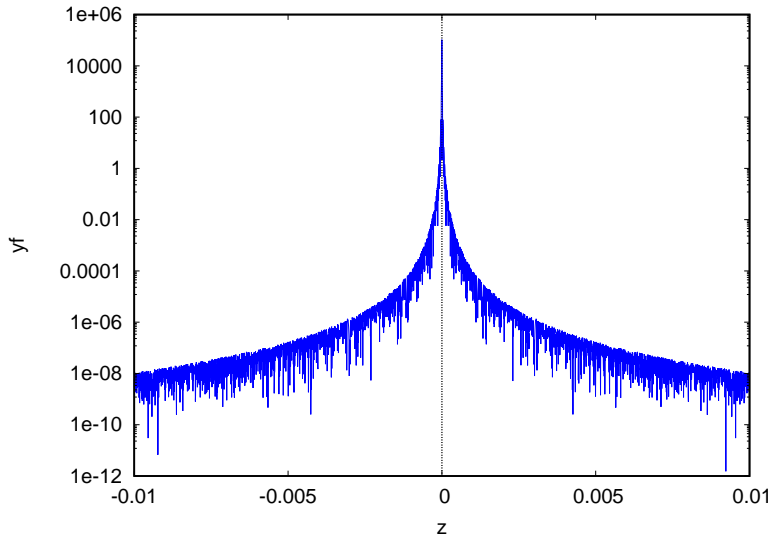
```
(%o23)  $-\left(\frac{1}{12}\right) + \frac{x^2}{180} - \frac{x^4}{6720} + \dots$ 
```

```
(%i 24) yf(z):=abs(float(y(z))-ev(''(taylor(y(x),x,0,4)),numer,x=z))
```

```
(%o24)  $yf(z) := \left| \text{float}(y(z)) - \text{ev}\left(-\left(\frac{1}{12}\right) + \frac{x^2}{180} - \frac{x^4}{6720} + \dots, \text{numer}, x = z\right) \right|$ 
```

```
(%i 27) tm_plot2d([yf],[z,-1e-2,1e-2],logy)
```

arithmetic error FLOATING-POINT-INVALID-OPERATION signalled
plot2d: expression evaluates to non-numeric value somewhere in plotting range.



(%o27) true

For expression E

$$E = \frac{B - 3C}{\theta^2} = \frac{\theta^2 - 3(\theta - \sin(\theta))}{\theta^5} \quad (5)$$

(%i 28) $y(x) := \frac{\frac{1 - \cos(x)}{x^2} - 3 \frac{1 - \frac{\sin(x)}{x}}{x^2}}{x^2}$

(%o28) $y(x) := \frac{\frac{1 - \cos(x)}{x^2} - 3 \left(\frac{1 - \frac{\sin(x)}{x}}{x^2} \right)}{x^2}$

(%i 29) `taylor(y(x), x, 0, 4)`

(%o29) $-\left(\frac{1}{60}\right) + \frac{x^2}{1260} - \frac{x^4}{60480} + \dots$

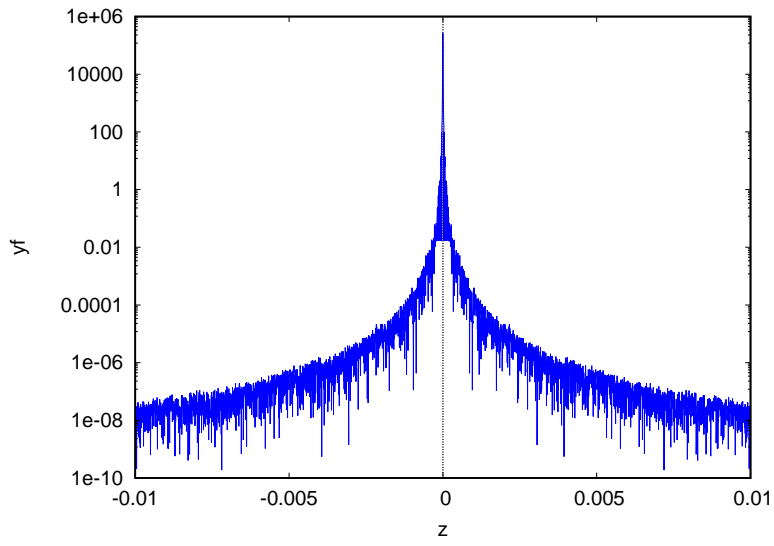
(%i 30) `yf(z) := abs(float(y(z)) - ev(''(taylor(y(x), x, 0, 4)), numer, x=z))`

(%o30) $yf(z) := \left| \text{float}(y(z)) - \text{ev} \left(-\left(\frac{1}{60}\right) + \frac{x^2}{1260} - \frac{x^4}{60480} + \dots, \text{numer}, x = z \right) \right|$

(%i 31) `tm_plot2d([yf], [z, -1e-2, 1e-2], logy)`

arithmetic error FLOATING-POINT-INVALID-OPERATION signalled

plot2d: expression evaluates to non-numeric value somewhere in plotting range.



(%o31) true

(%i 32)